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This is a summary of some of the author's recent study ([7], [8], [9]). Although we explain the essential idea fully in §1, it is desirable that readers should refer to [7], [8] and [9] for more details.

So far we have had only one method, which is the following (*), to obtain the topological equivalence for given two C^∞ map germs.

(*) For given two C^∞ map germs $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, take an appropriate one parameter family $F : (\mathbf{R}^n \times [0,1], \{0\} \times [0,1]) \to (\mathbf{R}^p, 0)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Then, prove that F is in fact topologically trivial.

Thus, it is significant to give an alternative systematic method for the topological classification even in a single $\mathcal{K}$-orbit, which is the purpose of this study.

A C^∞ deformation germ $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ of f is said to be *Thom trivial* (resp. *transversely Thom trivial*) if there exist C-regular stratifications in the sense of Bekka ([1]) $\mathcal{S}$ of $\mathbf{R}^n \times \mathbf{R}^k$, $\mathcal{T}$ of $\mathbf{R}^p \times \mathbf{R}^k$ and $\{\mathbf{R}^k\}$ of $\mathbf{R}^k$ such that the following (T1) and (T2) (resp. (T1), (T2) and (T3)) hold:

(T1) the map germ

$$(\Phi, \pi) : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p \times \mathbf{R}^k, (0,0))$$

is a Thom map germ with respect to $\mathcal{S}$ and $\mathcal{T}$.

(T2) the map germ

$$\pi' : (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^k, 0)$$

is a stratified map germ (or equivalently in this case, a Thom map germ) with respect to $\mathcal{T}$ and $\{\mathbf{R}^k\}$.

(T3) T_0 is transeverse to $\{0\} \times \mathbf{R}^k$ ($\subset \mathbf{R}^p \times \mathbf{R}^k$), where T_0 is the stratum of $\mathcal{T}$ which contains the origin $(0,0)$ of $\mathbf{R}^p \times \mathbf{R}^k$.

For given two C^∞ map germs $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, we consider the following four conditions.

(i) f and g are topologically equivalent.

(ii) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that the following (a), (b) and (c) are satisfied.

(a) $f(x) = M(x)g(s(x))$,

(b) The C^∞ map germ $F : (\mathbf{R}^n \times \mathbf{R}^p, (0,0)) \to (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a Thom trivial deformation germ of f,

(c) The C^∞ map germ $G : (\mathbf{R}^n \times \mathbf{R}^p, (0,0)) \to (\mathbf{R}^p, 0)$ given by

$$G(x, \lambda) = g(x) - M(s^{-1}(x))^{-1}\lambda$$

is a Thom trivial deformation germ of g.

(iii) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that (a), (b) of (ii) are satisfied.

(iv) There exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that the following (a), (b) are satisfied.

(a) $f(x) = M(x)g(s(x))$,

(b) The C^∞ map germ $F : (\mathbf{R}^n \times \mathbf{R}^p, (0,0)) \to (\mathbf{R}^p, 0)$ given by

$$F(x, \lambda) = f(x) - M(x)\lambda$$

is a transversely Thom trivial deformation germ of f.

A C^∞ map germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ is said to be *MT stable* if the jet extension of it is multi-transverse to the Thom-Mather canonical stratification of the jet space. As a consequence, every C^∞ deformation germ of f is Thom trivial (see [4], [6]]). Thus, the condition (ii) is a generalization of the assumption of the following well-known theorem ([3]).

Theorem 0. 1 (M. Fukuda and T. Fukuda) *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be MT stable map germs. Suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that*

$$f(x) = M(x)g(s(x)).$$

Then, they are topologically equivalent.

We can see that not only for MT-stable map germs but also for examples of Looijenga [5] and Damon [2] every C^∞ deformation of them are Thom trivial. Thus, the following theorem 0.2 realy generalizes theorem 0.1. Note that theorem 0.2 holds without any special assumptions.

Theorem 0. 2 ([8]) *The condition (ii) implies the condition (i).*

Next, we consider the condition (iii). Before giving the result, let us investigate one example.

Example Let $f(x,y) = (x, y^3 + xy)$, $g(x,y) = (x, y^3)$ and

$$M(x,y) = \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix}.$$

Then,

$$f(x,y) = M(x,y)g(x,y).$$

Since f is C^∞ stable, the deformation

$$F(x,y) = f(x,y) - M(x,y)\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

is C^∞ trivial, so Thom trivial because (2,2) is in the nice range.

However, it is easy to see that f and g are not topologically equivalent.

This example shows that the condition (iii) does not necessarily imply the topological equivalence. Nevertheless, we can show the following.

Theorem 0. 3 ([8]) *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map germs with rank zero. Then, the condition (iii) implies the condition (i).*

Although theorems 0.2 and 0.3 explain some topological structure in a $\mathcal{K}$-orbit and are interesting by themselves, unfortunately it is a little difficult to use them thoroughly. We would like to have results which are more easy to use. In order to answer our request, the condition (iv) was introduced.

Theorem 0. 4 ([9]) *The condition (iv) implies the condition (i).*

By using theorem 0.4, we can show the following.

Theorem 0. 5 ([9]) *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be an Thom-stable map germ. Suppose that $tf(\theta(n)) + f^*m_p\theta(f)$ contains $m_n^k\theta(f)$. Then f is topologically determined of order $2k$.*

Theorem 0.5 improves the following Gaffney's result.

Theorem 0. 6 ([10]) *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be an MT-stable map germ. Suppose that $tf(\theta(n)) + f^*m_p\theta(f)$ contains $m_n^k\theta(f)$. Then f is topologically determined of order $2k+1$.*

1 Strategy

In this section, we explain the essential idea of proofs of theorems 0.2, 0.3 and 0.4.

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be a C^∞ map germ and $\Phi : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p, 0)$ be a C^∞ deformation germ of f. Suppose that there exists a C^0 $\mathcal{A}$-morphism from Φ to f. Then, by the definition of C^0 $\mathcal{A}$- morphism, there exists a C^0 map germs $h : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^k, (0,0))$, $H : (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^p \times \mathbf{R}^k, (0,0))$ and $\varphi : (\mathbf{R}^k, 0) \to (\mathbf{R}^k, 0)$ such that the following (1.1) and (1.2) hold.

(1.1) For any representatives $\tilde{h}$ of h and $\widetilde{H}$ of H, there exist neighborhoods U of the origin in $\mathbf{R}^n$, V of the origin in $\mathbf{R}^k$ and W of the origin in $\mathbf{R}^p$ such that the restrictions $\tilde{h}|_{U\times\{\lambda\}}$ and $\widetilde{H}|_{W\times\{\lambda\}}$ are homeomorphisms for any $\lambda \in V$.

(1.2) The following diagram commutes..

$$\begin{array}{ccccc}
(\mathbf{R}^n \times \mathbf{R}^k, (0,0)) & \xrightarrow{(\Phi,\pi_\lambda)} & (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbf{R}^k, 0) \\
{\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle \varphi}\downarrow \\
(\mathbf{R}^n \times \mathbf{R}^k, (0,0)) & \xrightarrow{(f,\pi_\lambda)} & (\mathbf{R}^p \times \mathbf{R}^k, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbf{R}^k, 0)
\end{array}$$

By (1.2), we may write

$$h = (h_1, \varphi) \qquad \text{and} \qquad H = (H_1, \varphi).$$

Let $\varphi'_H : (\mathbf{R}^k, 0) \to (\mathbf{R}^p, 0)$ be the C^0 map germ given by

(1.3) $$\varphi'_H(\lambda) = H_1(0, \lambda).$$

The map germ (1.3) is the key in this study. We set also $h' : (\mathbf{R}^n \times \mathbf{R}^k, (0,0)) \to (\mathbf{R}^n \times \mathbf{R}^p, (0,0))$ as

$$h'(x, \lambda) = (h_1(x, \lambda), \varphi'_H(\lambda))$$

and $H' : (\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^p, (0,0,0)) \to (\mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^p, (0,0,0))$ as

$$H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - \varphi'_H(\lambda)).$$

Then we can show that $\{h', H', \varphi'_H\}$ is a C^0 $\mathcal{K}$-morphism from Φ to F, where F is the graph deformation of f given by $F(x, y) = f(x) - y$ (see [9]).

Returning to our situation, we let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be C^∞ map germs. We suppose that there exist a germ of C^∞ diffeomorphism $s : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and a C^∞ map germ $M : (\mathbf{R}^n, 0) \to (GL(p, \mathbf{R}), M(0))$ such that $f(x) = M(x)g(s(x))$. We concentrate on considering the following C^∞ deformation germ of f

(1.4) $$f(x) - M(x)\lambda.$$

This deformation germ is linear with respect to parameter variables. Remark that the parameter space of (1.4) is p-dimensional. Thus, if there exists a C^0 $\mathcal{A}$-morphism $\{h, H, \varphi\}$ from Φ to f, then the map germ (1.3) is a map germ between the same dimensional space.

Next, we suppose furthermore that the deformation germ (1.4) is C^0 trivial. Then, of course there exists a C^0 $\mathcal{A}$-morphism $\{h, H, \varphi\}$ from Φ to f. Thus, from the above argument we see that there exists a C^0 $\mathcal{K}$-morphism $\{h', H', \varphi'_H\}$ from (1.4) to the graph deformation F of f. In particular, we have the following equality.

$$f(h_1(x, g(s(x)))) = H_1(0, g(s(x)))$$

Finally, we can show the following (see [8]).

Lemma 1. 1 *If the map germ (1.3) is a germ of homeomorphism, then the endomorphism germ of* $(\mathbf{R}^n, 0)$ *given by*

$$x \mapsto h_1(x, g(s(x)))$$

is also a germ of homeomorphism.

Thus, we see that

Lemma 1. 2 *If the map germ (1.3) is a germ of homeomorphism, then f and g are topologically equivalent.*

By this strategy, theorems 0.2, 0.3 and 0.4 are proved.

参考文献

[1] K. Bekka, *C-régularité et trivialité topologique*, pp. 42-62 in "Singularity Theory and its Applications": Warwick 1989 Part I (eds. D. Mond and J. Montaldi), Lecture Notes in Math. **1462**, Springer, Berlin, 1991.

[2] J. Damon, *Finite determinacy and topological triviality II: Sufficient conditions and topological stability*, Compositio Mathematica **47**(1982), 101-132.

[3] M. Fukuda and T. Fukuda, *Algebras $Q(f)$ determine the topological types of generic map germs*, Inventiones math. **52**(1979), 231-237.

[4] C. G. Gibson et al., *Topological stability of smooth mappings*, Lecture Notes in Math. **552**, Springer, Berlin, 1976.

[5] E. J. N. Looijenga, *On the semi-universal deformation of a simple-elliptic singularity I : unimodularity*, Topology **16**(1977), 257-262.

[6] J. Mather, *How to stratify mappings and jet spaces* , Lecture Notes in Math. **535**, 128-176, Springer, Berlin, 1976.

[7] T. Nishimura, *The graph deformation of a* C^r *stable map germ is* C^r $\mathcal{K}$*-versal for* $0 \leq r \leq \infty$, preprint.

[8] T. Nishimura, *Topological equivalence of* $\mathcal{K}$-equivalent map germs, Journal of the London Matheomatical Society (to appear).

[9] T. Nishimura, *Topological equivalence of* $\mathcal{K}$-equivalent map germs *II*, in preparation.

[10] C. T. C. Wall, *Finite determinacy of smooth map germs*, Bull. London Math. Soc., **13**(1981), 481-539.

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